Regularities for distribution dependent SDEs with fractional noises

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Outline

The problem and related works

Distribution dependent stochastic differential equations (DDSDEs), also called McKean-Vlasov or mean-field SDEs, is of the form:

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t, \quad X_0 = \xi \in L^p(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P}).$$

where W is a Brownian motion and $\mathscr{L}_{X_t}$ denotes the law of X_t.

- F.-Y. Wang, *Distribution dependent SDEs for Landau type equations*, SPA, 2018.
- D. Baños, *The Bismut-Elworthy-Li formula for mean-field stochastic differential equations*, AIHP, 2018.
- P.P. Ren and F.-Y. Wang, *Bismut formula for Lions derivative of distribution dependent SDEs and applications*, JDE, 2019.
- M. Röckner and X.C. Zhang, *Well-posedness of distribution dependent SDEs with singular drifts*, Bernoulli, 2021.
- W. Liu, Y.L. Song, J.L. Zhai and T.S. Zhang, *Large and moderate deviation principles for McKean-Vlasov SDEs with jumps*, PA, 2022.
- X. Huang and F.-Y. Wang, *Regularities and exponential ergodicity in entropy for SDEs driven by distribution dependent noise*, arXiv:2209.14619.
- V. Barbu and M. Röckner, *Uniqueness for nonlinear Fokker-Planck equations and for McKean-Vlasov SDEs: The degenerate case*, JFA, 2023.

The problem and related works

DDSDEs driven by fractional Brownian motion (FBM) B^H with Hurst parameter $H \in (0,1)$:

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(\mathscr{L}_{X_t})\mathrm{d}B_t^H, \quad X_0 = \xi \in L^p(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P}).$$

- X.L. Fan, X. Huang, Y.Q. Suo and C.G. Yuan, *Distribution dependent SDEs driven by fractional Brownian motions*, SPA, 2022.
- X.L. Fan, T. Yu and C.G. Yuan, *Asymptotic behaviors for distribution dependent SDEs driven by fractional Brownian motions*, SPA, 2023.
- L. Galeati, F.A. Harang and A. Mayorcas, *Distribution dependent SDEs driven by additive fractional Brownian motion*, PTRF, 2023.
- G.J. Shen, J. Xiang and J.L. Wu, *Averaging principle for distribution dependent stochastic differential equations driven by fractional Brownian motion and standard Brownian motion*, JDE, 2022.

The problem and related works

Our concerned equation:

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t \mathrm{d}B_t^H + \tilde{\sigma}_t(\mathscr{L}_{X_t})\mathrm{d}\tilde{B}_t^{\tilde{H}}, \quad X_0 = \xi, \tag{1}$$

where $b : [0,T] \times \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^d, \sigma : [0,T] \to \mathbb{R}^d \otimes \mathbb{R}^d, \tilde{\sigma} : [0,T] \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d, \xi$ is an $\mathbb{R}^d$-valued random variable, and $B^H, \tilde{B}^{\tilde{H}}$ are respectively two independent FBMs with Hurst parameters $H \in (0,1)$ and $\tilde{H} \in (1/2, 1)$ independent of ξ, and the stochastic integral can be regarded as the Wiener integral.

- A d-FBM $(B_t^H)_{t\in[0,T]} = (B_t^{H,1}, \cdots, B_t^{H,d})_{t\in[0,T]}$ with $H \in (0,1)$ is a centered, H-self similar Gaussian process with the covariance function $\mathbb{E}(B_t^{H,i} B_s^{H,j}) = R_H(t,s)\delta_{i,j}$, where

$$R_H(t,s) := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \in [0,T].$$

- The FBM generalizes the standard Wiener process ($H = 1/2$) and has stationary increments. However, the increments are correlated with a power law correlation decay, which asserts the FBM is a non-Markovian process that is the dominant feature of equation (1).

The problem and related works

Our aim:

1. To prove the well-posedness of (1).
2. To investigate the regularity for (1).

For the second aim, we will study the regularity of the maps

$$\mu \mapsto P_t^* \mu, \quad t \in [0, T],$$

where $P_t^* \mu := \mathscr{L}_{X_t}$ for X_t solving (1) with initial distribution $\mathscr{L}_{X_0} = \mu \in \mathscr{P}_p(\mathbb{R}^d)$.

Observe that a probability measure is determined by integrals of $f \in \mathscr{B}_b(\mathbb{R}^d)$, it suffices to investigate the regularity of the functionals

$$\mu \mapsto (P_t f)(\mu) := \int_{\mathbb{R}^d} f \mathrm{d}(P_t^* \mu), \;\; f \in \mathscr{B}_b(\mathbb{R}^d), t \in [0, T].$$

More precisely, with regards to equation (1), we address the following question:

(Question) Under what conditions does the functional $P_t f$ have dimensional-free Harnack inequalities and Bismut formulas?

The problem and related works

Our motivation:

- D. Baños (AIHP, 2018) investigated the sensitivity of prices of options w.r.t. the initial value of the underlying asset price, and pointed out that the Bismut formula gives a better approximation of the sensitivity.
- The Harnack inequality may imply the gradient estimate and entropy estimate.
- X.L. Fan, X. Huang, Y.Q. Suo and C.G. Yuan (SPA, 2022) shown that for distribution-free noise ($\tilde{\sigma} = 0$ in equation (1), i.e. $\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t \mathrm{d}B_t^H, \quad X_0 = \xi$), Bismut formulas for $P_t f$ are established by using Malliavin calculus. However, for distribution dependent noise, these formulas are still open.

Well-posedness

DDSDE driven by two independent fractional Brownian motions B^H and $\tilde{B}^{\tilde{H}}$:

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t \mathrm{d}B_t^H + \tilde{\sigma}_t(\mathscr{L}_{X_t})\mathrm{d}\tilde{B}_t^{\tilde{H}}, \quad X_0 = \xi, \tag{2}$$

where $H \in (0,1), \tilde{H} \in (1/2, 1)$, $\xi \in L^p(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$ with $p \geq 1$, and the coefficients $b : [0,T] \times \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^d, \sigma : [0,T] \to \mathbb{R}^d \otimes \mathbb{R}^d, \tilde{\sigma} : [0,T] \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable functions.

(H1) There exists a non-decreasing function $\kappa_\cdot$ such that for every $t \in [0,T], x, y \in \mathbb{R}^d, \mu, \nu \in \mathscr{P}_p(\mathbb{R}^d)$,

$$|b_t(x,\mu) - b_t(y,\nu)| \leq \kappa_t(|x-y| + \mathbb{W}_p(\mu,\nu)), \quad \|\tilde{\sigma}_t(\mu) - \tilde{\sigma}_t(\nu)\| \leq \kappa_t \mathbb{W}_p(\mu,\nu),$$

and

$$|b_t(0,\delta_0)| + \|\sigma_t\| + \|\tilde{\sigma}_t(\delta_0)\| \leq \kappa_t.$$

For any $p \geq 1$, let $\mathcal{S}^p([0,T])$ be the space of $\mathbb{R}^d$-valued, continuous $(\mathscr{F}_t)_{t\in[0,T]}$-adapted processes ψ on $[0,T]$ satisfying

$$\|\psi\|_{\mathcal{S}^p} := \left(\mathbb{E} \sup_{t\in[0,T]} |\psi_t|^p\right)^{1/p} < \infty.$$

Well-posedness

Definition

A stochastic process $X = (X_t)_{0\le t\le T}$ on $\mathbb{R}^d$ is called a solution of (2), if $X \in \mathcal{S}^p([0,T])$ and $\mathbb{P}$-a.s.,

$$X_t = \xi + \int_0^t b_s(X_s, \mathscr{L}_{X_s})\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H + \int_0^t \tilde{\sigma}_s(\mathscr{L}_{X_s})\mathrm{d}\tilde{B}_s^{\tilde{H}}, \quad t \in [0,T].$$

- Note that $\sigma_\cdot$ and $\tilde{\sigma}_\cdot(\mathscr{L}_{X_\cdot})$ are both deterministic functions, then $\int_0^t \sigma_s \mathrm{d}B_s^H$ and $\int_0^t \tilde{\sigma}_s(\mathscr{L}_{X_s})\mathrm{d}\tilde{B}_s^{\tilde{H}}$ can be regarded as Wiener integrals w.r.t. fractional Brownian motions.

Theorem (Fan-Huang-Ling, arXiv:2304.00768)

Suppose that $\xi \in L^p(\Omega \to \mathbb{R}^d, \mathscr{F}_0, \mathbb{P})$ with $p \ge 1$ and one of the following conditions:

(I) $H \in (1/2, 1), b, \sigma, \tilde{\sigma}$ satisfy **(H1)** and $p > \max\{1/H, 1/\tilde{H}\}$;

(II) $H \in (0, 1/2), b, \tilde{\sigma}$ satisfies **(H1)**, σ_t does not depend on t and $p > 1/\tilde{H}$.

Then Eq. (2) has a unique solution $X \in \mathcal{S}^p([0,T])$. Moreover, let $(X_t^\mu)_{t\in[0,T]}$ be the solution to (2) with $\mathscr{L}_{X_0} = \mu \in \mathscr{P}_p(\mathbb{R}^d)$ and denote $P_t^*\mu = \mathscr{L}_{X_t^\mu}, t \in [0,T]$. Then it holds

$$\mathbb{W}_p(P_t^*\mu, P_t^*\nu) \le C_{p,T,\kappa,\tilde{H}} \mathbb{W}_p(\mu,\nu), \quad \mu,\nu \in \mathscr{P}_p(\mathbb{R}^d).$$

Well-posedness

Sketch of the proof

For any $\mu \in C([0,T], \mathscr{P}_p)$, consider

$$\mathrm{d}X_t = b_t(X_t, \mu_t)\mathrm{d}t + \sigma_t \mathrm{d}B_t^H + \tilde{\sigma}_t(\mu_t)\mathrm{d}\tilde{B}_t^{\tilde{H}}, \quad t \in [0,T], X_0 = \xi. \text{ (Denote its solution as } X_t^{\mu,\xi})$$

- To show $\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{\mu,\xi}|^p\right) < \infty$.
- Define the mapping $\Phi^\xi : C([0,T], \mathscr{P}_p(\mathbb{R}^d)) \to C([0,T], \mathscr{P}_p(\mathbb{R}^d))$ as

  $$\Phi_t^\xi(\mu) = \mathscr{L}_{X_t^{\mu,\xi}}, \quad t \in [0,T].$$

  To show

  $$\rho_{\lambda_0}(\Phi^\xi(\mu), \Phi^\xi(\nu)) < \frac{1}{2}\rho_{\lambda_0}(\mu,\nu), \quad \mu,\nu \in E^\xi,$$

  where λ_0 is a proper constant, and $E^\xi := \{\mu \in C([0,T]; \mathscr{P}_p(\mathbb{R}^d)) : \mu_0 = \mathscr{L}_\xi\}$ is equipped with the complete metric

  $$\rho_{\lambda_0}(\nu,\mu) := \sup_{t\in[0,T]} \mathrm{e}^{-\lambda_0 t}\mathbb{W}_p(\nu_t,\mu_t), \quad \mu,\nu \in E^\xi.$$
- Using the Banach fixed point theorem, we conclude that

  $$\Phi_t^\xi(\mu) = \mu_t, \quad t \in [0,T]$$

  has a unique solution $\mu \in E^\xi$.

Remark

- Main tool: (The Hardy-Littlewood inequality) Let $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. If $f : \mathbb{R}_+ \to \mathbb{R}$ belongs to $L^p(0,\infty)$, then $I^{\alpha}_{0+}f(x)$ converges absolutely for almost every x, and moreover

$$\|I^{\alpha}_{0+}f\|_{L^q(0,\infty)} \le C_{p,q}\|f\|_{L^p(0,\infty)}$$

holds for some positive constant $C_{p,q}$. Here, the left-sided fractional Riemann-Liouville integral of f of order α is defined as

$$I^{\alpha}_{0+}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(y)}{(x-y)^{1-\alpha}}\mathrm{d}y.$$

Remark

Under the same conditions as the theorem above, we obtain that for any $t \in [0,T]$,

$$\mathbb{E}\left(\sup_{s\in[0,t]} |\varrho^{\mu}_s - \varrho^{\nu}_s|^p\right) \le C_{p,T,\kappa,\tilde{H}} t^{p\tilde{H}} \mathbb{W}_p(\mu,\nu)^p.$$

Here we have set $\varrho^{\mu}_s := \int_0^s \tilde{\sigma}_r(P^*_r\mu)\mathrm{d}\tilde{B}^{\tilde{H}}_r$ for all $s \in [0,T]$ and $\mu \in \mathscr{P}(\mathbb{R}^d)$.

The non-degenerate case: Log-Harnack inequality

DDSDE:

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t \mathrm{d}B_t^H + \tilde{\sigma}_t(\mathscr{L}_{X_t})\mathrm{d}\tilde{B}_t^{\tilde{H}}, \ \ X_0 = \xi.$$

Assumptions

(H1') For every $t \in [0,T]$, $b_t(\cdot,\cdot) \in C^{1,(1,0)}(\mathbb{R}^d \times \mathscr{P}_p(\mathbb{R}^d))$. Moreover, there exists a non-decreasing function $\kappa.$ such that for any $t \in [0,T],\ x,y \in \mathbb{R}^d,\ \mu,\nu \in \mathscr{P}_p(\mathbb{R}^d)$,

$$\|\nabla b_t(\cdot,\mu)(x)\| + |D^L b_t(x,\cdot)(\mu)(y)| \le \kappa_t, \ \ \|\tilde{\sigma}_t(\mu) - \tilde{\sigma}_t(\nu)\| \le \kappa_t \mathbb{W}_p(\mu,\nu),$$

and $|b_t(0,\delta_0)| + \|\sigma_t\| + \|\tilde{\sigma}_t(\delta_0)\| \le \kappa_t$.

(H2) There exists a constant $\tilde{\kappa} > 0$ such that

(i) for any $t,s \in [0,T],\ x,y,z_1,z_2 \in \mathbb{R}^d,\ \mu,\nu \in \mathscr{P}_p(\mathbb{R}^d)$,

$$\begin{aligned}&\|\nabla b_t(\cdot,\mu)(x) - \nabla b_s(\cdot,\nu)(y)\| + |D^L b_t(x,\cdot)(\mu)(z_1) - D^L b_s(y,\cdot)(\nu)(z_2)| \\ &\le \tilde{\kappa}(|t-s|^\alpha + |x-y|^\beta + |z_1-z_2|^\gamma + \mathbb{W}_p(\mu,\nu)),\end{aligned}$$

where $\alpha \in (H-1/2, 1]$ and $\beta,\gamma \in (1-1/(2H), 1]$.

(ii) σ is invertible and σ^{-1} is Hölder continuous of order $\delta \in (H-1/2, 1]$:

$$\|\sigma^{-1}(t) - \sigma^{-1}(s)\| \le \tilde{\kappa}|t-s|^\delta, \ \ t,s \in [0,T].$$

The non-degenerate case: Log-Harnack inequality

Theorem (Fan-Huang-Ling, arXiv:2304.00768)

Consider Eq. (2). If one of the two following assumptions holds:

(I) $H \in (1/2, 1), b, \sigma, \tilde{\sigma}$ satisfy **(H1')**, **(H2)** and $p \geq 2(1+\beta)$;

(II) $H \in (0, 1/2), b, \tilde{\sigma}$ satisfies **(H1)**, σ_t does not depend on t and $p \geq 2$.

Then for any $t \in (0, T], \mu, \nu \in \mathscr{P}_p(\mathbb{R}^d)$ and $0 < f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$(P_t \log f)(\nu) \leq \log(P_t f)(\mu) + \varpi(H),$$

where

$$\varpi(H) = \begin{cases} C_{T,\kappa,\tilde{\kappa},H,\tilde{H}} \left(1 + \mathbb{W}_p(\mu,\nu)^{2\beta} + \frac{1}{t^{2H}}\right) \mathbb{W}_p(\mu,\nu)^2, & H \in (1/2, 1), \\ C_{T,\kappa,H,\tilde{H}} \left(1 + \frac{1}{t^{2H}}\right) \mathbb{W}_p(\mu,\nu)^2, & H \in (0, 1/2). \end{cases}$$

The non-degenerate case: Log-Harnack inequality

Remark

The log-Harnack inequality obtained above is equivalent to the following entropy-cost estimate

$$\mathrm{Ent}(P_t^*\nu|P_t^*\mu) \le \varpi(H), \quad t \in (0,T], \mu,\nu \in \mathscr{P}_p(\mathbb{R}^d),$$

where $\mathrm{Ent}(P_t^*\nu|P_t^*\mu)$ is the relative entropy of $P_t^*\nu$ with respect to $P_t^*\mu$ and p is given as in the theorem above.

Facts needed in the proof of the theorem:

- $\mathcal{H}$: the reproducing kernel Hilbert space
  $K_H^* : \mathcal{H} \to L^2([0,T],\mathbb{R}^d)$, $K_H : L^2([0,T],\mathbb{R}^d) \to I_{0+}^{H+1/2}(L^2([0,T],\mathbb{R}^d))$, $R_H = K_H \circ K_H^*$.
- W is a d-dimensional Wiener process such that

$$B_t^H = \int_0^t K_H(t,s)\mathrm{d}W_s, \quad t \in [0,T].$$

The non-degenerate case: Log-Harnack inequality

Sketch of the proof

- For fixed $t_0 \in (0,T]$, consider the following coupling DDSDE: $t \in [0,t_0]$,
$$\mathrm{d}Y_t = \left[b_t(X_t^\mu, P_t^*\mu) + \frac{1}{t_0}(X_0^\mu - X_0^\nu + \varrho_{t_0}^\mu - \varrho_{t_0}^\nu)\right]\mathrm{d}t + \sigma_t \mathrm{d}B_t^H + \tilde{\sigma}_t(P_t^*\nu)\mathrm{d}\tilde{B}_t^{\tilde{H}}, \quad Y_0 = X_0^\nu. \quad (3)$$
- Let $\bar{Y}_t = Y_t - \varrho_t^\nu$ and rewrite (3) as
$$\mathrm{d}\bar{Y}_t = b_t(\bar{Y}_t + \varrho_t^\nu, P_t^*\nu)\mathrm{d}t + \sigma_t \mathrm{d}\bar{B}_t^H, \quad t \in [0,t_0], \quad \bar{Y}_0 = Y_0 = X_0^\nu,$$
where
$$\bar{B}_t^H := B_t^H - \int_0^t \sigma_s^{-1}\zeta_s \mathrm{d}s = \int_0^t K_H(t,s)\left(\mathrm{d}W_s - K_H^{-1}\left(\int_0^\cdot \sigma_r^{-1}\zeta_r \mathrm{d}r\right)(s)\mathrm{d}s\right)$$
with
$$\zeta_s := b_s(Y_s, P_s^*\nu) - b_s(X_s^\mu, P_s^*\mu) - \frac{1}{t_0}(X_0^\mu - X_0^\nu + \varrho_{t_0}^\mu - \varrho_{t_0}^\nu).$$
- $\mathscr{L}_{\bar{Y}_{t_0}}|_{R^{\tilde{H},0}\mathrm{d}\mathbb{P}^{\tilde{H},0}} = \mathscr{L}_{\bar{X}_{t_0}^\nu}|_{\mathbb{P}^{\tilde{H},0}}$, where $\bar{X}_\cdot^\nu := X_\cdot^\nu - \varrho_\cdot^\nu$ satisfies SDE
$$\mathrm{d}\bar{X}_t^\nu = b_t(\bar{X}_t^\nu + \varrho_t^\nu, P_t^*\nu)\mathrm{d}t + \sigma_t \mathrm{d}B_t^H, \quad t \in [0,t_0], \quad \bar{X}_0^\nu = X_0^\nu.$$

  Then, the law of $Y_{t_0} = \bar{Y}_{t_0} + \varrho_{t_0}^\nu$ under $R^{\tilde{H},0}\mathrm{d}\mathbb{P}^{\tilde{H},0}$ is the same as one of $X_{t_0}^\nu = \bar{X}_{t_0}^\nu + \varrho_{t_0}^\nu$ under $\mathbb{P}^{\tilde{H},0}$.

The non-degenerate case: Bismut formula

Bismut formula for the L-derivative of (2):

For every $t \in (0,T], \mu \in \mathscr{P}_p(\mathbb{R}^d)$ and $\phi \in L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, we are to find an integrable random variable $M_t(\mu,\phi)$ such that

$$D^L_\phi(P_tf)(\mu) = \mathbb{E}\left(f(X_t^\mu)M_t(\mu,\phi)\right), \ \ f \in \mathscr{B}_b(\mathbb{R}^d).$$

For any $\mu \in \mathscr{P}_p(\mathbb{R}^d)$, let $(X_t^\mu)_{t\in[0,T]}$ be the solution to (2) with $\mathscr{L}_{X_0^\mu} = \mu$ and $P_t^*\mu = \mathscr{L}_{X_t^\mu}$ for every $t \in [0,T]$.
For any $\varepsilon \in [0,1]$ and $\phi \in L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, let $X_t^{\mu_{\varepsilon,\phi}}$ denote the solution of (2) with $X_0^{\mu_{\varepsilon,\phi}} = (\mathrm{Id}+\varepsilon\phi)(X_0^\mu), \mu_{\varepsilon,\phi} := \mathscr{L}_{(\mathrm{Id}+\varepsilon\phi)(X_0^\mu)}$.

Introduce the spatial derivative of X_t^μ along ϕ:

$$\nabla_\phi X_t^\mu := \lim_{\varepsilon\to 0}\frac{X_t^{\mu_{\varepsilon,\phi}} - X_t^\mu}{\varepsilon}, \ \ t \in [0,T], \ \phi \in L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu).$$

Assumptions

(H3) There exists a non-decreasing function $\kappa_\cdot$ such that

$$|D^L\tilde{\sigma}_t(\mu)(x)| \le \kappa_t, \ \ t \in [0,T], \ x \in \mathbb{R}^d, \mu \in \mathscr{P}_p(\mathbb{R}^d).$$

The non-degenerate case: Bismut formula

Lemma

Assume that **(H1')**, **(H3)** hold and σ_t does not depend on t if $H \in (0, 1/2)$. For any $\mu \in \mathscr{P}_p(\mathbb{R}^d)$ and $\phi \in L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ with $p > \max\{1/H, 1/\tilde{H}\}$ if $H \in (1/2, 1)$ or $p > 1/\tilde{H}$ if $H \in (0, 1/2)$, then the following assertions hold.

(i) $\nabla_\phi X^\mu_\cdot$ exists in $L^p(\Omega \to C([0,T]; \mathbb{R}^d), \mathbb{P})$ such that $\nabla_\phi X^\mu_\cdot$ is the unique solution of the following linear SDE

$$\begin{aligned} \mathrm{d}G_t^\phi &= \Big[\nabla_{G_t^\phi} b_t(\cdot, \mathscr{L}_{X_t^\mu})(X_t^\mu) + \Big(\mathbb{E}\langle D^L b_t(y, \cdot)(\mathscr{L}_{X_t^\mu})(X_t^\mu), G_t^\phi\rangle\Big)\,|_{y=X_t^\mu}\Big]\,\mathrm{d}t \\ &\quad + \mathbb{E}\langle D^L \tilde{\sigma}_t(\mathscr{L}_{X_t^\mu})(X_t^\mu), G_t^\phi\rangle \mathrm{d}\tilde{B}_t^{\tilde{H}}, \quad G_0^\phi = \phi(X_0^\mu), \end{aligned}$$

and

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |\nabla_\phi X_t^\mu|^p\Big) \le C_{p,T,\kappa,H,\tilde{H}} \|\phi\|^p_{L^p(\mu)}.$$

(ii) It holds

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left(\sup_{s\in[0,t]}\left|\frac{\varrho_s^{\mu_\varepsilon,\phi} - \varrho_s^\mu}{\varepsilon} - \Lambda_s\right|^p\right) = 0,$$

where $\Lambda_\cdot$ is defined as

$$\Lambda_s := \int_0^s \Big\langle \mathbb{E}[\langle D^L \tilde{\sigma}_r(P_r^*\mu)(X_r^\mu), \nabla_\phi X_r^\mu\rangle], \mathrm{d}\tilde{B}_r^{\tilde{H}}\Big\rangle, \quad s \in [0, T].$$

The non-degenerate case: Bismut formula

Theorem (Fan-Huang-Ling, arXiv:2304.00768)

Consider Eq. (2). If one of the two following assumptions holds:

(I) $H \in (1/2, 1), b, \sigma, \tilde{\sigma}$ satisfy **(H1')**, **(H2)** and **(H3)**;

(II) $H \in (0, 1/2), b, \tilde{\sigma}$ satisfies **(H1')**, **(H3)** and σ_t does not depend on t,

then for any $t \in (0, T], f \in \mathscr{B}_b(\mathbb{R}^d), \mu \in \mathscr{P}_p(\mathbb{R}^d)$ and $\phi \in L^p(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ with $p \geq 2(1+\beta)$ if $H \in (1/2, 1)$ or $p \geq 2$ if $H \in (0, 1/2)$, $D^L_\phi(P_t f)(\mu)$ exists and satisfies

$$D^L_\phi(P_t f)(\mu) = \mathbb{E}\left(f(X_t^\mu) \int_0^t \left\langle K_H^{-1}\left(\int_0^\cdot \sigma_r^{-1} \Upsilon_{r,t} \mathrm{d}r\right)(s), \mathrm{d}W_s \right\rangle \right), \tag{4}$$

where $\Upsilon_{\cdot,\cdot}$ is given by

$$\begin{aligned}\Upsilon_{r,t} = &\frac{\phi(X_0^\mu) + \Lambda_t}{t} + \nabla b_r(\cdot, P_r^*\mu)(X_r^\mu)\left(\frac{t-r}{t}\phi(X_0^\mu) - \frac{r}{t}\Lambda_t + \Lambda_r\right) \\ &+ \mathbb{E}[\langle D^L b_r(x, \cdot)(P_r^*\mu)(X_r^\mu), \nabla_\phi X_r^\mu \rangle]|_{x = X_r^\mu}, \quad 0 \leq r < t \leq T\end{aligned}$$

with $\Lambda_\cdot$ defined in the lemma above.

The non-degenerate case: Bismut formula

Remark

(i) The term $K_H^{-1}\left(\int_0^\cdot \sigma_r^{-1}\Upsilon_{r,t}\mathrm{d}r\right)(s)$ on the right-hand side of (4) can rewrite as follows

$$K_H^{-1}\left(\int_0^\cdot \sigma_r^{-1}\Upsilon_{r,t}\mathrm{d}r\right)(s)=\begin{cases}\frac{(H-\frac12)s^{H-\frac12}}{\Gamma(\frac32-H)}\Bigg[\frac{s^{1-2H}\sigma_s^{-1}\Upsilon_{s,t}}{H-\frac12}+\sigma_s^{-1}\Upsilon_{s,t}\int_0^s\frac{s^{\frac12-H}-r^{\frac12-H}}{(s-r)^{\frac12+H}}\mathrm{d}r+ \\ \Upsilon_{s,t}\int_0^s\frac{(\sigma_s^{-1}-\sigma_r^{-1})r^{\frac12-H}}{(s-r)^{\frac12+H}}\mathrm{d}r+\int_0^s\frac{(\Upsilon_{s,t}-\Upsilon_{r,t})\sigma_r^{-1}r^{\frac12-H}}{(s-r)^{\frac12+H}}\mathrm{d}r\Bigg], & H\in(\frac12,1),\\ \frac{\sigma^{-1}s^{H-\frac12}}{\Gamma(\frac12-H)}\int_0^s\frac{r^{\frac12-H}\Upsilon_{r,t}}{(s-r)^{\frac12+H}}\mathrm{d}r, & H\in(0,\frac12).\end{cases}$$

(ii) The estimate of the L-derivative: for any $t\in(0,T], f\in\mathscr{B}_b(\mathbb{R}^d), \mu\in\mathscr{P}_p(\mathbb{R}^d)$,

$$\|D^L(P_tf)(\mu)\|_{L_\mu^{p^*}}\leq C_{T,\kappa,\tilde\kappa,H,\tilde H}\left(1+\frac{1}{t^H}\right)\left((P_t|f|^{p^*})(\mu)\right)^{\frac{1}{p^*}},$$

where $C_{T,\kappa,\tilde\kappa,H,\tilde H}$ is a positive constant which is independent of $\tilde\kappa$ when $H\in(0,1/2)$, and $p\geq 2(1+\beta)$ if $H\in(1/2,1)$ or $p\geq 2$ if $H\in(0,1/2)$.

The degenerate case: Log-Harnack inequality

Let A and B be two matrices of order $m \times m$ and $m \times l$, we now consider the following distribution dependent degenerate SDE:

$$\begin{cases} \mathrm{d}X_t^{(1)} = (AX_t^{(1)} + BX_t^{(2)})\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t \mathrm{d}B_t^H + \tilde{\sigma}_t(\mathscr{L}_{X_t})\mathrm{d}\tilde{B}_t^{\tilde{H}}, \end{cases} \tag{5}$$

where $X_t = (X_t^{(1)}, X_t^{(2)}), b : [0, T] \times \mathbb{R}^{m+l} \times \mathscr{P}_p(\mathbb{R}^{m+l}) \to \mathbb{R}^l, \sigma(t)$ is an invertible $l \times l$-matrix for every $t \in [0, T]$, $\tilde{\sigma} : [0, T] \times \mathscr{P}_p(\mathbb{R}^{m+l}) \to \mathbb{R}^l \otimes \mathbb{R}^l$ are measurable.

To establish the log-Harnack inequality, we let

$$U_t = \int_0^t \frac{s(t-s)}{t^2} \mathrm{e}^{-sA} BB^* \mathrm{e}^{-sA^*} \mathrm{d}s \geq \ell(t) \mathrm{I}_{m \times m}, \ t \in (0, T], \tag{6}$$

where $\ell \in C([0, T])$ satisfies $\ell(t) > 0$ for any $t \in (0, T]$ and $\mathrm{I}_{m \times m}$ is the $m \times m$ identity matrix. It is obvious that U_t is invertible with $\|U_t^{-1}\| \leq 1/\ell(t)$ for every $t \in (0, T]$.

The degenerate case: Log-Harnack inequality

Theorem (Fan-Huang-Ling, arXiv:2304.00768)

Consider Eq. (5). Assume (6) and if one of the two following assumptions holds:

(I) $H \in (1/2, 1), b, \sigma, \tilde{\sigma}$ satisfy **(H1')** and **(H2)** with $d = m + l$, and $p \geq 2(1+\beta)$;

(II) $H \in (0, 1/2), b, \tilde{\sigma}$ satisfies **(H1)** with $d = m + l$, σ_t does not depend on t and $p \geq 2$.

Then for any $t \in (0, T]$, $\mu, \nu \in \mathscr{P}_p(\mathbb{R}^{m+l})$ and $0 < f \in \mathscr{B}_b(\mathbb{R}^{m+l})$,

$$(P_t \log f)(\nu) \leq \log(P_t f)(\mu) + \chi(H),$$

where

$$\chi(H) = \begin{cases} C_{T,\kappa,\tilde{\kappa},H,\tilde{H}} \left(1 + \mathbb{W}_p(\mu,\nu)^{2\beta} + \frac{1}{t^{2H}} + \frac{1}{\ell^2(t)} + \frac{1}{t^{2H}\ell^2(t)}\right) \mathbb{W}_p(\mu,\nu)^2, & H \in (1/2, 1), \\ C_{T,\kappa,H,\tilde{H}} \left(1 + \frac{1}{t^{2H}} + \frac{1}{\ell^2(t)} + \frac{1}{t^{2H}\ell^2(t)}\right) \mathbb{W}_p(\mu,\nu)^2, & H \in (0, 1/2). \end{cases}$$

The degenerate case: Bismut formula

Theorem (Fan-Huang-Ling, arXiv:2304.00768)

Consider Eq. (5). Assume (6) and if one of the two following assumptions holds:

(I) $H \in (1/2, 1), b, \sigma, \tilde{\sigma}$ satisfy **(H1')**, **(H2)** and **(H3)**;

(II) $H \in (0, 1/2), b, \tilde{\sigma}$ satisfies **(H1')**, **(H3)** with $d = m + l$, σ_t does not depend on t,

then for any $t \in (0, T], f \in \mathscr{B}_b(\mathbb{R}^{m+l}), \phi \in L^p(\mathbb{R}^{m+l} \to \mathbb{R}^{m+l}, \mu)$ and $\mu \in \mathscr{P}_p(\mathbb{R}^{m+l})$ with $p \geq 2(1+\beta)$ if $H \in (1/2, 1)$ or $p \geq 2$ if $H \in (0, 1/2)$, $D^L_\phi(P_T f)(\mu)$ exists and satisfies

$$D^L_\phi(P_t f)(\mu) = \mathbb{E}\left(f(X^\mu_t) \int_0^t \left\langle K_H^{-1}\left(\int_0^\cdot \sigma_r^{-1}\Theta_{r,t}\mathrm{d}r\right)(s), \mathrm{d}W_s \right\rangle \right),$$

where $\Theta_{\cdot,\cdot}$ is defined as

$$\Theta_{s,t} = \nabla b_s(\cdot, P_s^*\mu)(X_s^\mu)\hbar_{s,t} + \mathbb{E}[\langle D^L b_s(x,\cdot)(P_s^*\mu)(X_s^\mu), \nabla_\phi X_s^\mu\rangle]|_{x=X_s^\mu} - (\Xi_t)'(s).$$

with

$$\hbar_{s,t} := \left(\mathrm{e}^{sA}\phi^{(1)}(X_0^\mu) + \int_0^s \mathrm{e}^{(s-r)A}B\left(\phi^{(2)}(X_0^\mu) + \Xi_t(r) + \Lambda_r\right)\mathrm{d}r, \phi^{(2)}(X_0^\mu) + \Xi_t(s) + \Lambda_s\right),$$

$$\Xi_t(s) := -\frac{s}{t_0}(\phi^{(2)}(X_0^\mu) + \Lambda_t) - \frac{s(t-s)}{t^2}B^*\mathrm{e}^{-sA^*}U_t^{-1}\phi^{(1)}(X_0^\mu)$$
$$- \frac{s(t-s)}{t^2}B^*\mathrm{e}^{-sA^*}U_t^{-1}\int_0^{t_0}\mathrm{e}^{-rA}B\left[\frac{t-r}{t_0}\phi^{(2)}(X_0^\mu) - \frac{r}{t_0}\Lambda_t + \Lambda_r\right]\mathrm{d}r.$$

The degenerate case: Bismut formula

Remark

The entropy-cost and intrinsic derivative estimates:

For any $t \in (0,T]$, $\mu, \nu \in \mathscr{P}_p(\mathbb{R}^d)$ and $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$\mathrm{Ent}(P_t^* \nu | P_t^* \mu) \leq \chi(H)$$

and

$$\|D^L(P_t f)(\mu)\|_{L^{p^*}_\mu} \leq C_{T,\kappa,\tilde{\kappa},H,\tilde{H}} \left(1 + \frac{1}{t^H} + \frac{1}{\ell(t)} + \frac{1}{t^H \ell(t)}\right) \left((P_t |f|^{p^*})(\mu)\right)^{\frac{1}{p^*}}$$

where $C_{T,\kappa,\tilde{\kappa},H,\tilde{H}}$ is a positive constant which is independent of $\tilde{\kappa}$ when $H \in (0, 1/2)$, and $p \geq 2(1+\beta)$ if $H \in (1/2, 1)$ or $p \geq 2$ if $H \in (0, 1/2)$.

To guarantee (6) holds, one needs to impose some non-degeneracy condition on the matrix B. For instance, assume the following Kalman rank condition:

$$\mathrm{Rank}[B, AB, \cdots, A^k B] = m \tag{7}$$

holds for some integer number $k \in [0, m-1]$ (in particular, if $k = 0$, (7) reduces to $\mathrm{Rank}[B] = m$), then (6) is satisfied with $\ell(t) = C(t \wedge 1)^{2k+1}$ for positive constant C.

Further problems and main references

Further problems

- Well-posedness in multiplicative case
- Chaos propogation

Main references

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Thank you very much for your kind attention!